Effects of elliptic oblateness on harmonic oscillator in elliptic paraboloid potential

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ABSTRACT

The harmonic oscillator is an important and typical physical model in quantum mechanics and quantum optics. It is very important and widely used, and has been confirmed by the development and application of science and technology. The harmonic oscillator in the elliptic paraboloid potential is studied and effects of the elliptic oblateness on the harmonic oscillator in the elliptic paraboloid potential are revealed. The energy of the harmonic oscillator in the elliptical paraboloid potential is quantized, which is described by two quantum numbers *n,* and *m*. Generally speaking, the maximum value of the probability density peak decreases as the extremum number of the probability density increases. However, this reduction is with oscillations and fluctuations, which shows even a maximum structure for the smaller quantum number. For the elliptic paraboloid potential, the spatial distribution of the probability density on different cutting surfaces is various. The flatter the ellipse is, the greater the probability density of the ellipse center, and the smaller the extreme of the edge peak of the probability density will be.

Keywords: Elliptic paraboloid, infinite potential well, harmonic oscillator, elliptic oblateness

1. INTRODUCTION

The harmonic oscillator is an important and typical physical model in quantum mechanics and quantum optics. It is very important and widely used, and has been confirmed by the development and application of science and technology. In particular, we can further analyze and discuss the problems related to the hydrogen atom through the relationship between the spatial harmonic oscillator and the hydrogen atom. In recent years, using the double wave function method, asymptotic iteration method, the Fourier transform method, and so on, some researchers study the problem of the one-dimensional or isotropous quantum harmonic oscillator, in which significant results have been obtained [1-14]. In this paper, by the method of separating variables, the steady-state Schrodinger equation of a potential well of an infinite elliptic parabola is solved, and quantum properties of the harmonic oscillator in the potential well of an infinite elliptic parabola are analyzed and studied. The layout of this paper is as follows. In Section 2, using the method of separating variables, we derive solutions for a harmonic oscillator in a potential well of the infinite elliptic paraboloid. In Section 3, we discuss the wave function and the energy spectrum of the harmonic oscillator, revealing its interesting physical properties. Section 4 is summary and conclusion.

2. SOLUTIONS FOR THE SCHRÖDINGER EQUATION OF A HARMONIC OSCILLATOR IN A POTENTIAL WELL OF THE INFINITE ELLIPTIC PARABOLOID

There is an infinite deep potential well of an ellipse paraboloid, as depicted in Figure 1, the potential function versus the coordinates read as

$$u(x,y)=\frac{1}{2}k_1x^2+\frac{1}{2}k_2y^2=\frac{x^2}{2a^2}+\frac{y^2}{2b^2} \tag{1}$$

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where $k_1 = \frac{1}{a^2}$, and $k_2 = \frac{1}{b^2}$ are potential energy constants of the harmonic oscillator in different directions, when $u(x,y) = u_1$, on the plane of $u(x,y) = u_1$, we get the corresponding elliptic equation of the potential function

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1 \tag{2}$$

with $a' = \sqrt{2u_1}a = \sqrt{\frac{2u_1}{k_1}}$, and $b' = \sqrt{\frac{2u_1}{k_2}}$.

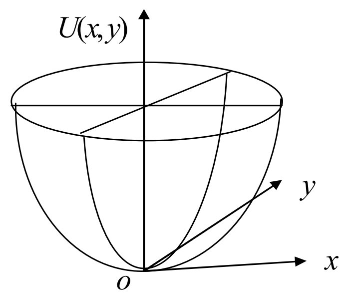

Figure 1. A potential well of the infinite elliptic paraboloid.

On the xou plane of $y = 0$, we get the corresponding parabola equation of the potential function

$$u = \frac{x^2}{a'^2} \tag{3}$$

On the plane $y = tg\theta\, x$, we get the corresponding parabola equation of the potential function

$$u = (\frac{1}{2}k_1 + \frac{1}{2}k_2 tg\theta)\, x^2 \tag{4}$$

The Stationary Schrödinger equation for the elliptic paraboloid potential is given by

$$[-\frac{\hbar^2}{2\mu}\nabla^2 + (\frac{k_1}{2}x^2 + \frac{k_2}{2}y^2)]\psi(x,y) = E\psi(x,y) \tag{5}$$

Considering $\psi(x,y) = f(x)\varphi(y)$, and plugging it into equation (5), we have

$$[-\frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial x^2} f(x)\varphi(y) - \frac{\hbar^2}{2\mu}\frac{\partial^2}{\partial y^2} f(x)\varphi(y) + (\frac{x^2}{a^2} + \frac{y^2}{b^2}) f(x)\varphi(y) = Ef(x)\varphi(y) \tag{6}$$

By separating variables, from equation (6), we get

$$[-\frac{\hbar^2}{2\mu}\frac{d^2}{dx^2} + \frac{1}{2}k_1 x^2] f(x) = E_1 f(x) \tag{7}$$

and

$$[-\frac{\hbar^2}{2\mu}\frac{d^2}{dy^2}+\frac{k_2}{2}y^2]\varphi(y)=E_2\varphi(y) \tag{8}$$

with $E = E_1 + E_2$.

Defining $\omega_1 = (\frac{k_1}{\mu})^{\frac{1}{2}}$, $\alpha_1 = (\frac{\mu\omega_1}{\hbar})^{\frac{1}{2}}$, $\lambda_1 = \frac{2E_1}{\hbar\omega_1}$, $\omega_2 = (\frac{k_2}{\mu})^{\frac{1}{2}}$, $\alpha_2 = (\frac{\mu\omega_2}{\hbar})^{\frac{1}{2}}$, $\xi{=}\alpha x$, and $\lambda_2 = \frac{2E_2}{\hbar\omega_2}$, and solving equation (7), we obtain the stationary wavefunction of equation (7)

$$f_n(x) = C_n e^{-\frac{\xi_1^2}{2}} H_n(\xi_1) = C_n e^{-\frac{\alpha_1^2 x^2}{2}} H_n(\alpha_1 x) \tag{9}$$

$$E_n = \hbar\omega_1(n+\frac{1}{2}), \quad (n = 0,1,2...) \tag{10}$$

where $C_n = (\frac{\alpha_1}{\sqrt{\pi}2^n n!})^{1/2}(n = 0,1,2...)$, and $H_n(\xi_1)$ is a Hermitian polynomial.

Similarly, solving equation (8), we obtain the corresponding stationary wavefunction

$$\varphi_n(y) = C_m e^{-\frac{\xi_2^2}{2}} H_m(\xi_2) = C_m e^{-\frac{\alpha_2^2 y^2}{2}} H_m(\alpha_2 y) \tag{11}$$

$$E_m = \hbar\omega_2(m+\frac{1}{2}), \quad (m = 0,1,2...) \tag{12}$$

where $C_m = (\frac{\alpha_2}{\sqrt{\pi}2^m m!})^{1/2}(m = 0,1,2...)$, and $H_m(\xi_2)$ is a Hermitian polynomial.

Then the stationary wavefunction for the elliptic paraboloid potential is given by

$$\begin{aligned}\psi_{n\,m}(x,y) &= C_n C_m e^{-\frac{\alpha_1^2 x^2+\alpha_2^2 y^2}{2}} H_n(\alpha_1 x) H_m(\alpha_2 y) \\ &= (\frac{\alpha_1\alpha_2}{\pi 2^{n+m} n!m!})^{1/2} e^{-\frac{\alpha_1^2 x^2+\alpha_2^2 y^2}{2}} H_n(\alpha_1 x) H_m(\alpha_2 y)\end{aligned} \tag{13}$$

and the total energy of the system is given by

$$E_{n\,m} = \hbar[\omega_1(n+\frac{1}{2})+\omega_2(m+\frac{1}{2})], \quad (n = 0,1,2..., \quad m = 0,1,2...) \tag{14}$$

3. ENERGY SPECTRUM AND PROBABILITY DENSITY DISTRIBUTION OF THE HARMONIC OSCILLATOR IN AN ELLIPTIC PARABOLOID POTENTIAL

From equation (14), we see that comparison with the one-dimensional linear harmonic oscillator, energy spectrums of the harmonic oscillator of an elliptic paraboloid are richer.

The energy spectrum of the system has two quantum numbers n and *m*. When $n = 0$, and $m = 0$, we get the ground state energy of the harmonic oscillator of an elliptic paraboloid

$$E_{\mathbf{0}\,\mathbf{0}} = \hbar[\frac{1}{2}\omega_1 + \frac{1}{2}\omega_2)] = \frac{1}{2\sqrt{\mu}}\hbar(\sqrt{k_1}+\sqrt{k_2}) \tag{15}$$

Through the odd and even properties of Hermitic polynomials, the parity of the harmonic oscillator of an elliptic paraboloid it can be discussed.

From equation (13), we have

$$\begin{aligned}\psi_{n\,m}(-x,-y) &= C_n C_m e^{-\frac{\alpha_1^2 x^2+\alpha_2^2 y^2}{2}} H_n(-\alpha_1 x) H_m(-\alpha_2 y) \\ &= (-1)^{n+m} C_n C_m e^{-\frac{\alpha_1^2 x^2+\alpha_2^2 y^2}{2}} H_n(\alpha_1 x) H_m(\alpha_2 y) = (-1)^{n+m}\psi_{n\,m}(x,y)\end{aligned} \tag{16}$$

When $n+m$ is even, wavefunction exhibits even parity; $n+m$ is odd, it has odd parity.

The probability density function of the harmonic oscillator of an elliptic paraboloid is given by

$$|\psi_{n\,m}(x,y)|^2 = C_n^2 C_m^2 e^{-(\alpha_1^2 x^2+\alpha_2^2 y^2)} H_n^2(\alpha_1 x) H_m^2(\alpha_2 y) \tag{17}$$

Utilizing equation (17), one can present the distribution diagrams of the probability density of the harmonic oscillator of an elliptic paraboloid.

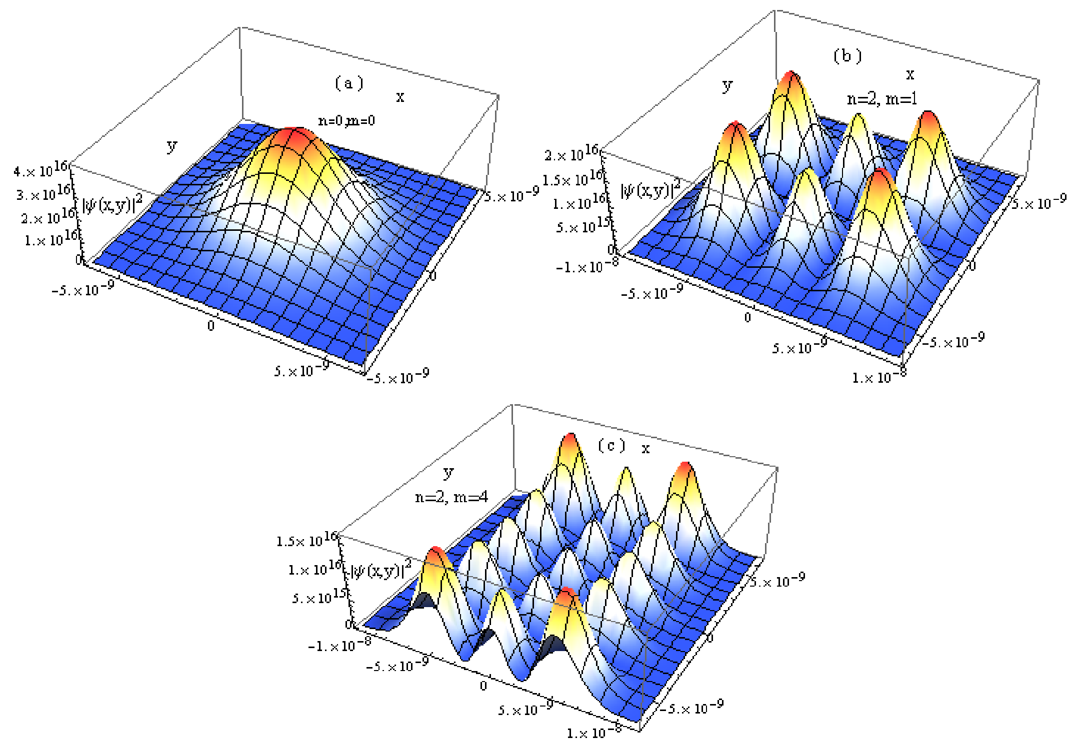

Figure 2. Probability density distribution as a function of variables *x* and *y* for different quantum numbers *n* and *m*.

Figure 2 presents probability density distributions of the harmonic oscillator in an elliptic paraboloid potential for different quantum numbers *n* and *m*, and exhibits effects of the quantum numbers on probability density distributions.

In Figure 2a, the quantum numbers are $n=0$, and $m=0$, the harmonic oscillator is in the ground state, the energy of the ground state $E_{0\,0} = \hbar[\omega_1+\omega_2)]/2$, and the probability density exhibits one maximum at the potential well center, where the probability of the oscillator appearance is the greatest. In Figure 2b, the quantum numbers are $n=2$, and $m=1$, its energy is $E_{2,1} = \hbar[5\omega_1+3\omega_2)]/2$, and the probability density exhibits six extremums, at the locations of which the probabilities of the oscillator appearance are relatively large. In Figure 2c, the quantum numbers are $n=2$, and $m=4$, its energy is $E_{2,1} = \hbar[5\omega_1 + 9\omega_2)]/2$, and the probability density exhibits fifteen extreme values at the locations of which the harmonic oscillator is most likely to occur.

From Figure 2, we see that numbers of extreme values of the probability density distribution satisfy $(n+1)(m+1)$.

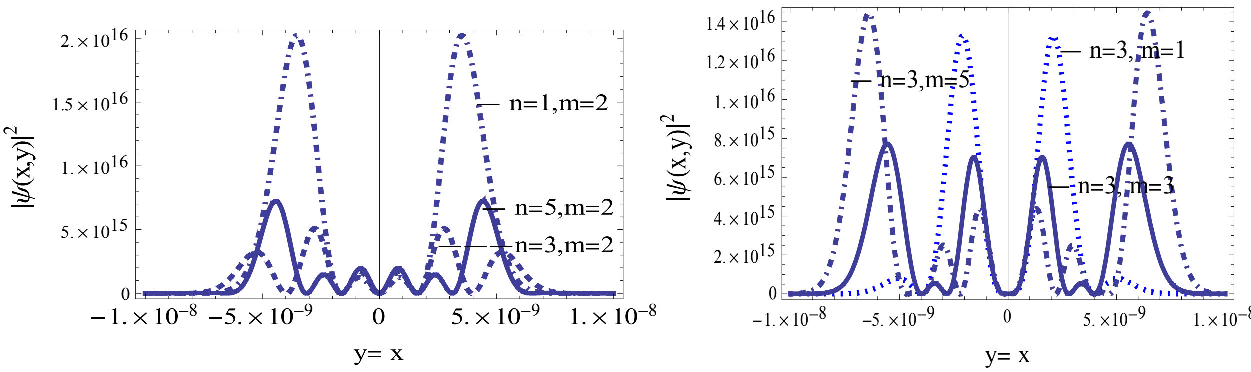

Figure 3. Effects of quantum numbers n, and m on the probability density.

In Figure 3a, the quantum number n is fixed to be $n=3$, the quantum number m is from 1, 3, to 5. From 1 to 3, the peak value of the probability density decreases; however, from 3 to 5, the peak value increases to the maximum.

In Figure 3b, the quantum number m is fixed to be $m=2$, the quantum number m changes from 1, 3, to 5. The peak value of the probability density changes similarly.

The quantum numbers n and m have regulatory effects on the peak value and the distribution of the probability density of the harmonic oscillator in the elliptic paraboloid potential.

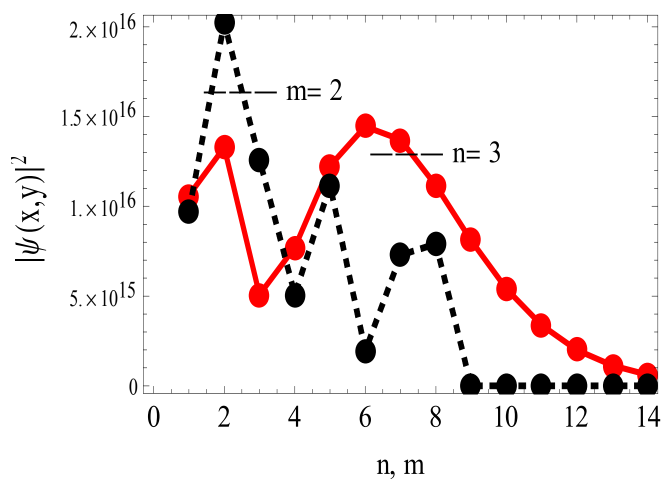

Figure 4. Effects of quantum numbers n, and m on the maximum of the probability density.

In order to further analyze the influence of the quantum number on probability density distribution, we exhibit effects of the quantum numbers n and m on the maximum value of the probability density in Figure 4. When one quantum number is fixed and another quantum number increases, the general trend is that the maximum number of the probability density increases and the maximum value of probability density decreases. However, this reduction is with oscillations and fluctuations, which shows even a maximum structure for the smaller quantum numbers.

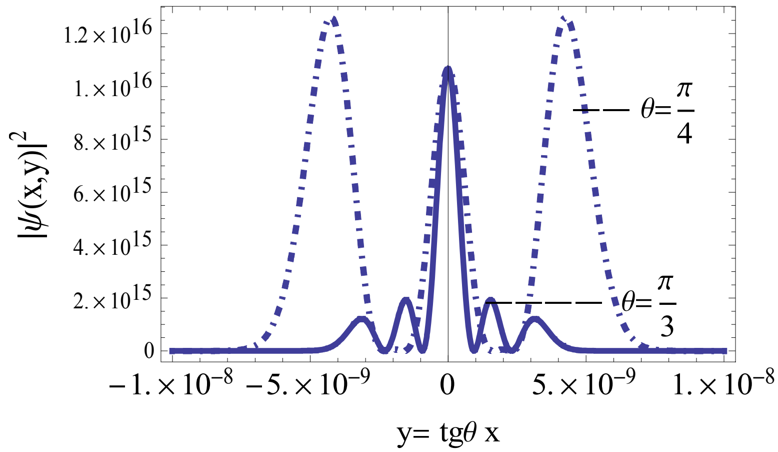

Figure 5. Probability density distributions for different section planes.

Under the same system parameters, the probability density distribution on different cutting surfaces is various. The plane $\theta = \pi/4$ compared with the plane $\theta = \pi/3$, as Figure 5 shown, the distribution of the probability density value is relatively strong on the plane $\theta = \pi/4$, especially, the maximum value of the edge peak being more prominent.

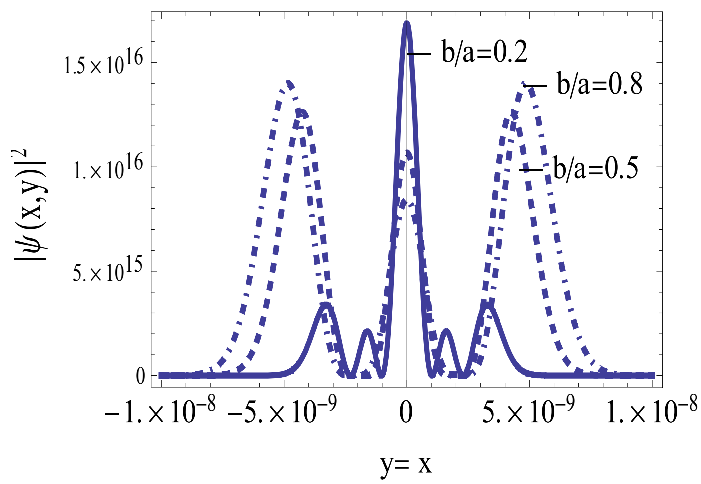

Figure 6. Effects of the elliptic oblateness on the probability density distribution.

Figure 6 shows the effects of the elliptic oblateness on the distribution of the probability density of the harmonic oscillator in the elliptic paraboloid potential.

The flatter the ellipse is, the greater the probability density of the ellipse center, and the smaller the extreme of the edge peak of the probability density will be.

4. SUMMARY AND CONCLUSION

From the above discussion, we derive the following main results.

The energy of the harmonic oscillator in the elliptic paraboloid potential is quantized, which is described by two quantum numbers *n*, and *m*.

The distribution of the probability density of the harmonic oscillator shows the extremum structure. The numbers of extreme values of the probability density distribution corresponding to $E_{n\,m} = \hbar[\omega_1(n+1/2)+\omega_2(m+1/2)]$ are $(n+1)(m+1)$. The quantum numbers *n* and *m* play a vital role in the maximum peak value of the probability density.

Generally speaking, the maximum peak value of the probability density decreases as the extremum number of the probability density increases. However, this reduction is with oscillations and fluctuations, which shows even a maximum structure for the smaller quantum numbers.

For the elliptic paraboloid potential, the spatial distribution of the probability density on different cutting surfaces is various. The flatter the ellipse is, the greater the probability density of the ellipse center, and the smaller the extreme of the edge peak of the probability density will be.

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