Chapter 1

Rationality in Progressive Transmission

1.1 Introduction

A progressive transmission scheme prioritizes the code bits according to their reduction in distortion (e.g., Ref. [92]). If the original image is I, the coding is actually done to $C = \Omega(I)$, where Ω represents a unitary hierarchical subband transformation (Ref. [4]). The 2D array $C = \{c_{i,j}\}$ has the same dimensions of I, and each element $c_{i,j}$ is called a transform coefficient at coordinate (i, j), which for the purpose of coding can be treated as an integer. In a progressive transmission scheme, the decoder initially sets the reconstruction vector $\hat{C}$ to zero and updates its components according to the coded message. After receiving the value (approximate or exact) of some coefficients, the decoder can obtain a reconstructed image $\hat{I} = \Omega^{-1}(\hat{C})$.

In an embedded wavelet scheme for progressive transmission (e.g., Refs. [68] and [97]), a tree structure, called a spatial orientation tree (SOT) in Ref. [92], naturally defines the spatial relationship on the hierarchical pyramid. Figure 1.1 illustrates how an SOT is defined in a pyramid constructed with recursive four-subband splitting. Each node of the tree corresponds to a pixel, and its direct descendants (offspring) correspond to the pixels of the same spatial orientation in the next finer level of the pyramid. Transform coefficients over an SOT correspond to a particular local spatial region of the original image, and thus, each SOT is associated with one spatial region, as illustrated in Fig. 1.1.

The embedded coding is distinctive from conventional coding in the sense that any SOT is coded bit-plane by bit-plane through successive-

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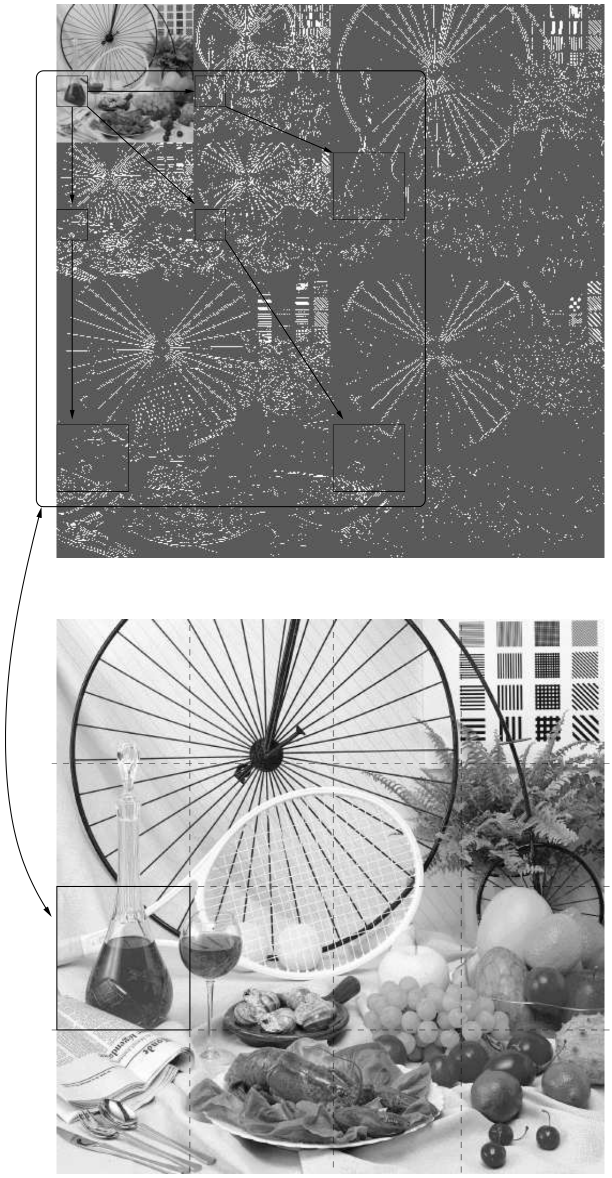

Figure 1.1 Each SOT in the hierarchical pyramid corresponds to a spatial region of the original image.

approximation quantization (Appendix A). Given an initial threshold T_i with $i = 0$, the successive approximation consists of an iteration over two scans, called the sorting pass and the refinement pass. The bit stream of the sorting pass, BS_{2i}, is generated using the set-partitioning approach (Ref. [92]), to locate the significant wavelet coefficients with respect to the threshold. The bit stream of the refinement pass, called BS_{2i+1}, results from the refinement of all the coefficients of the SOT that have not been quantized to zero so far. After a refinement pass, a new threshold may be computed as $T_i = T_{i-1}/2$ (with $i \geq 1$), where i is the iteration number. The algorithm may be iterated by applying successive sorting and refinement bit streams, BS_{2i} and BS_{2i+1}. The result of the coding of the particular SOT is a completely embedded bit stream as given by a number of sorting and refinement bit streams $BS_1\, BS_2 \cdots BS_K$.

This first chapter addresses the problem of SOT selection in the context of embedded image coding for progressive transmission. It extends the set partitioning in hierarchical trees (SPIHT) in Ref. [92] to consider the ordering of the trees, such that at every time instance, it always chooses the tree that provides the maximum expected increase in utility per coding bit. The approach is closely related to rate-constrained wavelet-based embedded image coding, since it is intended to show that an embedded coder can be optimized by means of a rational strategy for rate control by organizing the progressive transmission with expected increase in utility per coding bit. We denote such coding strategy as rational embedded wavelet image coding (REWIC).

Wavelet transform captures local characteristics in both space and frequency domains. Depending on the image content and the particular truncation time, some trees have higher coding priority than others in terms of expected increase in utility per coding bit. Therefore, it may be worthwhile to explore the ordering of the trees. The resultant coding bit stream can be truncated at any point and still maintain an optimal utility-per-coding-bit performance at every truncation point. Of course, the key issue then is the comparison of trees in terms of image quality improvement per bit. Utility functions are addressed in Sec. 1.4.

The overall objective of developing a rational approach to choosing among SOTs—presented in the form of a series of axioms as follows—does not purport to describe the ways in which individuals actually do behave in making choices among SOTs for their progressive transmission. At any truncation time, a prioritization system should choose without any outside knowledge among alternative SOTs *in such a way as to avoid certain forms of behavioral inconsistency.* The axioms are proposed simply to prescribe constraints that seem to us imperative to acknowledge in the image transmission problem.

This chapter demonstrates that with several plausible assumptions we may restrict the form of the utility function for ordering preferences in progressive transmission. We conclude that *some rational transmission systems may then exhibit aversion to risk* with respect to "gambles" on SOT-dependent quality of encoding, *while others favor taking such risks.*

1.2 Basic elements of the selection problem

We describe any situation in which choices are to be made, at truncation time t, among available SOTs for their transmission by successive-approximation quantization in terms of a decision problem whose structure is determined by three basic elements:

1. A set $\{R_i, i \in I\}$ of available SOTs, one of which is to be selected for transmission of a number of bit streams $S(R_i, t) = BS_j\, BS_{j+1} \cdots BS_{j+n}$ at truncation time t;

2. For each SOT R_i, a set $\{G_{l,i}; l \in L\}$ describing the gray-level occurrences in the spatial region that is reconstructed using the bit streams $S(R_i, t)$ candidates to be transmitted at time t, in addition to bit streams transmitted, before the time t, for R_i; where $G_{l,i}$ denotes the uncertain event pixel X within a spatial region associated with R_i, taking gray-level value l;

3. Corresponding to each set $\{G_{l,i}; l \in L\}$, a set of consequences $\{c_{l,i}; l \in L\}$ that induces the transmission of a number of bit streams $S(R_i, t) = BS_j\, BS_{j+1} \cdots BS_{j+n}$ for the particular SOT R_i.

The idea is as follows. Suppose we choose SOT R_i for transmission of a number of its bit streams $BS_j\, BS_{j+1} \cdots BS_{j+n}$, at truncation time t; then one set $\{G_{l,i}; l \in L\}$ occurs and leads to the corresponding consequence set $\{c_{l,i}; l \in L\}$. Thus, the choice of a R_i that is required at any truncation time t produces an outcome (the corresponding set of gray-level occurrences) that is beyond our control and induces a particular set of consequences (i.e., perceived visual fidelity of decoded outcome). The entire transmission of all bit planes for each SOT involves sequential considerations, but this may essentially reduce to repeated analyses based on the above structure.

It is clear that to choose a particular SOT R_i is to opt for the uncertain scenario labelled by the pair set $\{(G_{l,i}, c_{l,i}); l \in L\}$. Obviously, the perception of the state of uncertainty resulting from the choice of any particular SOT for further transmission is very much dependent on the information currently transmitted. Further information being transmitted will change the gray-level occurrences that result from selecting transmission R_i. Thus, the above representation captures the structure of a decision problem as perceived at a particular truncation time.

In addition to representing the structure of the selection of an SOT at truncation time t as a decision problem using the three elements discussed above, we must also represent the idea of preference as applied to the comparison of all of the pairs of available SOTs. We shall therefore need to consider a fourth basic element in the SOT selection problem, which is reformulated in terms of a decision problem: The relation $\preceq$, which expresses the preferences between pairs of available SOTs at a particular truncation time, so that $R_i \preceq R_j$ signifies that R_i is not preferred, at the truncation time, to R_j for further transmission of a number of bit streams.

We now give a formal definition of the selection of an SOT for transmission at a truncation time as a decision problem:

Definition 1.1: SOT selection as a decision problem. *The decision problem of SOT selection at the truncation time t is defined by four elements $\{\mathcal{R}, \mathcal{G}, \mathcal{C}, \preceq\}$, where:*

(i) $\mathcal{R}$ is the set of available SOTs;

(ii) $\mathcal{G}$ is the class of any possible set of gray-level occurrences corresponding to the transmission for a particular SOT;

(iii) $\mathcal{C}$ is the class of any set of possible consequences (perceived visual fidelity) associated with a gray-level occurrence set by means of the selection of an SOT;

(iv) $\preceq$ is a preference order between SOTs taking the form of a binary relation between SOTs in $\mathcal{R}$.

1.3 Basic axioms for avoiding forms of behavioral inconsistency

The operational notion of preference between SOTs, formalized by the binary relation $\preceq$, provides a qualitative basis for comparing SOTs. A number of coherence axioms are proposed that provide a minimal set of rules to ensure that qualitative comparisons based on $\preceq$ cannot have intuitively undesirable implications.

The first postulate states the essence of what is required for an orderly and systematic approach to comparing among SOTs: (a) if all consequences were equivalent, there would not be a decision problem; and (b) if the system aspires to make a rational choice between alternative SOTs, then it must at least be willing to express preferences between different SOTs.

Postulate 1.1

(i) Not all the consequences in $\mathcal{C}$ are equivalent; and

(ii) The transmission system is able to compare any pair of options concerning the selection of an SOT at truncation time t.

The second axiom is intended to impose rules of coherence on preference orderings that will exclude the possibility of two types of inconsistencies: First, the system prefers one SOT over another identical SOT; second, the system is willing to suffer the certain loss of something of value, which happens if $R_i \preceq R_j$, $R_j \preceq R_k$, and $R_k \preceq R_i$.

Postulate 1.2

(i) $R_i \preceq R_i$; *and*

(ii) If $R_i \preceq R_j$ *and* $R_j \preceq R_k$, *then* $R_i \preceq R_k$.

The binary relation $\preceq$ may also provide a qualitative basis for comparing, by extension, consequences and gray-level occurrence events. And the third axiom shall ensure the consistency of any kind of preferences (e.g., between consequences or gray-level occurrence events).

Postulate 1.3

(i) Preferences between visual fidelity of decoded outcomes (i.e., consequences) should not be affected by the transmission of further information;

(ii) If a gray-level occurrence set $\{G_{l,i}; l \in L\}$ *is more likely to relate to better decoded outcomes than another gray-level occurrence set* $\{G_{l,j}; l \in L\}$ *then the selection of the* SOT *that produces the former set is preferred to that producing the latter;*

(iii) If R_i *is preferred to* R_j *under the occurrence of event* G*, then comparison of options* R_i *and* R_j *(which are identical in preference if a different event occurs) depends entirely on consideration of what happens if* G *occurs.*

Postulates 1.1 to 1.3 then provide a minimal set of rules to ensure that qualitative comparisons based on the preference $\preceq$ cannot have intuitively undesirable implications. But we also need to introduce some form of quantification by setting up a standard unit of measurement that enables the transmission system to assign a numerical value to any given SOT in the selection problem. In short, precision through quantification is achieved by introducing some form of numerical standard into the system already equipped with a coherent qualitative ordering relation (Postulates 1.1 through 1.3). We shall regard it as essential to be able to aspire to some kind of quantitative precision in the context of comparing SOTs. It is therefore necessary that we have available some form of standard SOTs. This notion of quantization is given by means of two additional axioms (Postulates 1.4 and 1.5).

Postulate 1.4 *In the transmission system, there exists some form of standard SOTs, which will play a role analogous to the standard units of measurement.*

Postulate 1.5 *The standard family of SOTs provides a continuous scale against which any consequence or event can be precisely compared.*

Next, a pair of Propositions 1.1(a) and 1.1(b) serve to determine how numerical measures can be assigned to two of the elements of the SOT selection problem in the form of probabilities for gray-level occurrence events and utilities for consequences.

Proposition 1.1 *Any transmission system that aspires to analyze the SOT selection problem* $\{\mathcal{R}, \mathcal{G}, \mathcal{C}, \preceq\}$ *at any truncation time t in accordance with Postulates 1.1 through 1.5 should verify that*

(a) Degrees of belief about gray-level occurrence sets $\{G_{l,i}; l \in L\}$ *are represented in the form of finite probability distributions*

$$\{p(G_{l,i} \mid S(R_i, t)); l \in L\},$$

where $S(R_i, t) = BS_j \, BS_{j+1} \cdots BS_{j+n}$ *is the candidate bit stream to be transmitted at time t for* R_i*, and with* $p(G_{l,i} \mid S(R_i, t))$ *denoting the probability of gray-level l in the spatial region associated with SOT* R_i *and which was reconstructed using* $S(R_i, t)$ *in addition to the bit streams transmitted before time t;*

(b) Numerical values attached to the consequences $\{c_{l,i}; l \in L\}$ *foreseen if a particular SOT* R_i *is taken are represented in the form of a utility function.*

Proof.

(a) Proposition 1.1(a) states that, to avoid certain forms of behavioral inconsistency when a transmission system chooses, at any truncation time, among available SOTs for transmission, the gray-level occurrence sets $\mathcal{G}_i \equiv \{G_{l,i}; l \in L\}$ should be represented by probability distributions $\mathcal{P}_i \equiv \{p(G_{l,i} \mid S(R_i, t)); l \in L\}$. In such a framework, the actions available to the system are the various probability distributions $\mathcal{P}_i$ over $\mathcal{G}_i$, the latter constituting the gray-level occurrence set corresponding to each SOT selection. And this result directly comes from Proposition 2.11 in Ref. [10], which establishes formally that coherent, quantitative measures of uncertainty about events must take the form of probabilities: (i) coherent, quantitative degrees of belief have the structure of a finitely additive probability measure; moreover, (ii) significant events, i.e., events which are practically possible but not certain, should be assigned probability values in the open interval $(0, 1)$.

(b) Proposition 1.1(b) asserts that options in the SOT selection cannot be ordered without a specification of utilities (numerical values) for the consequences. Assuming a definition of utility that only involves comparison among consequences and options constructed with standard events, we would expect the utility of a consequence to be uniquely defined and to remain unchanged as new information is obtained, since the preference patterns among consequences is unaffected by additional information. This is indeed the case, as Proposition 2.21 in Ref. [10] establishes the decision problems for which extreme consequences are assumed to exist. In our problem it is attractive to have available the possibility, for conceptual and mathematical convenience, of dealing with sets of consequences not possessing extreme elements. But Proposition 2.23 in Ref. [10] also extends Proposition 2.21 in Ref. [10] to a more general situation in which extreme consequences are not assumed to exist.

Thus, in the following section we complete the specification of this decision problem by inducing the preference ordering through the introduction of a particular form of utility function $u(\cdot)$, which describes the numerical value $u(\mathcal{P}_i, G_{l,i})$, where the set of consequences $\mathcal{C}$ consists of all pairs $(\mathcal{P}_i, G_{l,i})$ representing the conjunctions of probability distributions and actual gray-level occurrences.

1.4 Progressive transmission utility functions

This section presents the basic axiomatic characterization of a utility function. The objective is twofold: firstly, to characterize the utility function with a minimal number of properties which are natural and thus desirable; and secondly, to determine the form of all utility functions satisfying these properties that we have stated to be desirable for progressive transmission.

The following postulate states the assumption that the statistical characterization of the decoded output corresponding to information transmitted for R_i up to this time is independent of the characterization of the decoded output corresponding to another SOT R_j.

Postulate 1.6 *Let R_i and R_j be any pair of SOTs. Preferences for decoded outcomes at truncation time t involving the two SOTs R_i and R_j depend only on the probability distributions that characterize reconstructions using the information transmitted for R_i and R_j, respectively, up to this time t, and not on their joint probability distribution.*