Chapter 7
Decomposition of Hadamard Matrices

We have seen in Chapter 1 that Hadamard's original construction of Hadamard matrices states that the Kronecker product of Hadamard matrices of orders m and n is a Hadamard matrix of order mn. The multiplicative theorem was proposed in 1981 by Agaian and Sarukhanyan [1] (see also [2]). They demonstrated how to multiply Williamson–Hadamard matrices in order to get a Williamson–Hadamard matrix of order $mn/2$. This result has been extended by the following:

- Craigen *et al*. [3]. They show how to multiply four Hadamard matrices of orders m, n, p, q in order to get a Hadamard matrix of order $mnpq/16$.
- Agaian [2] and Sarukhanyan *et al*. [4] show how to multiply several Hadamard matrices of orders $n_i, i = 1, 2, \ldots, k+1,$ to get a Hadamard matrix of order $(n_1 n_2 \cdots n_{k+1})/2^k$, k = 1, 2,…. They obtained a similar result for $A(n,k)$–type Hadamard matrices and for Baumert–Hall, Plotkin, and Geothals–Seidel arrays [5].
- Seberry and Yamada investigated the multiplicative theorem of Hadamard matrices of the generalized quaternion type using the M-structure [6].
- Phoong and Chang [7] show that the Agaian and Sarukhanyan theorem results can be generalized to the case of antipodal paraunitary (APU) matrices. A matrix function $H(z)$ is said to be paraunitary (PU) if it is unitary for all values of the parameters z, $H(z)H^T(1/z) = nI_n$ $n \geq 2$. One attractive feature of these matrices is their energy preservation properties that can avoid the noise or error amplification problem. For further details of PU matrices and their applications, we refer the reader to [8–10]. A PU matrix is said to be

an APU matrix if all of its coefficient matrices have ± 1 as their entries. For the special case of constant (memory less) matrices, APU matrices reduce to the well-known Hadamard matrices.

The analysis of the above-stated results relates with solution of the following problem.

Problem 1 [2, 11]: Let X_i and $A_i, i = 1, 2, \ldots, k$ be $(0, \pm 1)$ and (+1,–1) matrices of dimensions $p_1 \times p_2$ and $q_1 \times q_2$, respectively, and $p_1 q_1 = p_2 q_2 = n \equiv 0 \pmod 4$.

(a) What conditions must matrices X_i and A_i satisfy for

$$H = \sum_{i=1}^{k} X_i \otimes A_i, \tag{7.1}$$

to be a Hadamard matrix of order *n*, and

(b) How are these matrices constructed?

In this chapter, we develop methods for constructing matrices X_i and A_i, making it possible to construct new Hadamard matrices and orthogonal arrays. We also present a classification of Hadamard matrices based on their decomposability by orthogonal (+1,–1)-vectors. We will present multiplicative theorems of construction of a new class of Hadamard matrices and Baumert-Hall, Plotkin, and Geothals-Seidel arrays. Particularly, we will show that if there be *k* Hadamard matrices of order $m_1, m_2, \ldots, m_k$, then a Hadamard matrix of order $(m_1 m_2 \cdots m_k)/2^{k+1}$ exists. As an application of multiplicative theorems, one may find an example in [12–14].

7.1 Decomposition of Hadamard Matrices by (+1,–1) Vectors

In this section, a particular case of the problem given above is studied, i.e., the case when A_i is (+1,–1)-vectors.

Theorem 7.1.1: For matrix *H* [see (7.1)] to be an Hadamard matrix of order *n*, it is necessary and sufficient that there be $(0, \pm 1)$ matrices X_i and (+1,–1) matrices $A_i, i = 1, 2, \ldots, k$ of dimensions $p_1 \times p_2$ and $q_1 \times q_2$, respectively, satisfying the following conditions:

1. $p_1 q_1 = p_2 q_2 = n \equiv 0 (\mathrm{mod}\ 4),$
2. $X_i * X_j = 0, \quad i \neq j, \quad i, j = 1, 2, \cdots, k, \quad$ * is Hadamard product,
3. $\sum_{i=1}^{k} X_i \quad is\ (+1, -1) - \text{matrix},$
4. $\sum_{i=1}^{k} X_i X_i^T \otimes A_i A_i^T + \sum_{i,j=1}^{k} X_i X_j^T \otimes A_i A_j^T = nI_n, \quad i \neq j,$
5. $\sum_{i=1}^{k} X_i^T X_i \otimes A_i^T A_i + \sum_{i,j=1}^{k} X_i^T X_j \otimes A_i^T A_j = nI_n, \quad i \neq j.$

The first three conditions are evident. The two last conditions are jointly equivalent to conditions

$$HH^T = H^T H = nI_n. \tag{7.2}$$

Now, let us consider the case where A_i are (+1,–1) vectors. Note that any Hadamard matrix H_n of order *n* can be represented as

$$\begin{aligned} &a) \quad H_n = (++) \otimes X + (+-) \otimes Y, \\ &b) \quad H_n = \sum_{i=1}^{8} v_i \otimes A_i, \end{aligned} \tag{7.3}$$

where *X*, *Y* are $(0, \pm 1)$ matrices of dimension $n \times (n/2)$, A_i are $(0, \pm 1)$ matrices of dimension $n \times (n/4)$, and v_i are the following four-dimensional (+1,–1) vectors:

$$\begin{aligned} &v_1 = (++++), \quad v_2 = (++--), \quad v_3 = (+--+), \quad v_4 = (+-+-), \\ &v_5 = (+---), \quad v_6 = (+-++), \quad v_7 = (+++-), \quad v_8 = (++-+). \end{aligned} \tag{7.4}$$

Here, we give the examples of decomposition of the following Hadamard matrices:

$$H_4 = \begin{pmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{pmatrix}, \quad H_8 = \begin{pmatrix} + & + & + & + & + & + & + & + \\ + & - & + & - & + & - & + & - \\ + & + & - & - & + & + & - & - \\ + & - & - & + & + & - & - & + \\ + & + & + & + & - & - & - & - \\ + & - & + & - & - & + & - & + \\ + & + & - & - & - & - & + & + \\ + & - & - & + & - & + & + & - \end{pmatrix}. \tag{7.5}$$

We use the following notations:

$$\begin{gathered} w_1 = (++),\ w_2 = (+-),\ \ v_1 = (++++), \\ v_2 = (+-+-),\ \ v_3 = (++--),\ v_4 = (+--+). \end{gathered} \tag{7.6}$$

Example 7.1.1: The Hadamard matrix H_4 and H_8 can be decomposed as follows:

(1) Via two vectors:

$$H_4 = w_1 \otimes \begin{pmatrix} + & + \\ 0 & 0 \\ + & - \\ 0 & 0 \end{pmatrix} + w_2 \otimes \begin{pmatrix} 0 & 0 \\ + & + \\ 0 & 0 \\ + & - \end{pmatrix},$$

$$H_8 = w_1 \otimes \begin{pmatrix} + & + & + & + \\ 0 & 0 & 0 & 0 \\ + & - & + & - \\ 0 & 0 & 0 & 0 \\ + & + & - & - \\ 0 & 0 & 0 & 0 \\ + & - & - & + \\ 0 & 0 & 0 & 0 \end{pmatrix} + w_2 \otimes \begin{pmatrix} 0 & 0 & 0 & 0 \\ + & + & + & + \\ 0 & 0 & 0 & 0 \\ + & - & + & - \\ 0 & 0 & 0 & 0 \\ + & + & - & - \\ 0 & 0 & 0 & 0 \\ + & - & - & + \end{pmatrix}. \tag{7.7}$$

(2) Via four vectors:

$$H_4 = v_1 \otimes \begin{pmatrix} + \\ 0 \\ 0 \\ 0 \end{pmatrix} + v_2 \otimes \begin{pmatrix} 0 \\ + \\ 0 \\ 0 \end{pmatrix} + v_3 \otimes \begin{pmatrix} 0 \\ 0 \\ + \\ 0 \end{pmatrix} + v_4 \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ + \end{pmatrix},$$

$$H_8 = v_1 \otimes \begin{pmatrix} + & + \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ + & - \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + v_2 \otimes \begin{pmatrix} 0 & 0 \\ + & + \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ + & - \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + v_3 \otimes \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ + & + \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ + & - \\ 0 & 0 \end{pmatrix} + v_4 \otimes \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ + & + \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ + & - \end{pmatrix}. \tag{7.8}$$

Now, let us introduce matrices.

$$\begin{aligned} B_1 &= A_1 + A_2 + A_7 + A_8, \qquad & B_2 &= A_3 + A_4 + A_5 + A_6, \\ B_3 &= A_1 - A_2 - A_5 + A_6, \qquad & B_4 &= -A_3 + A_4 + A_7 - A_8. \end{aligned} \tag{7.9}$$

Theorem 7.1.1 [15]: For the existence of Hadamard matrices of order *n*, the existence of $(0,\pm1)$ matrices $B_i, i = 1,2,3,4$ of dimension $n \times (n/4)$ is necessary and sufficient, satisfying the following conditions:

$$\begin{aligned} &1.\quad B_1 * B_2 = 0, \quad B_3 * B_4 = 0, \\ &2.\quad B_1 \pm B_2, \quad B_3 \pm B_4 \quad \text{are } (+1,-1)-\text{matrices}, \\ &3.\quad \sum_{i=1}^{4} B_i B_i^T = \tfrac{n}{2} I_n, \\ &4.\quad B_i^T B_j = 0, \quad i \neq j, \quad i,j = 1,2,3,4, \\ &5.\quad B_i^T B_i = \tfrac{n}{2} I_{n/4}, \quad i,j = 1,2,3,4. \end{aligned} \tag{7.10}$$

Proof:

Necessity: Let H_n be a Hadamard matrix of order *n*. According to (7.1), we have

$$H_n = v_1 \otimes A_1 + v_2 \otimes A_2 + \cdots + v_8 \otimes A_8. \tag{7.11}$$

From this representation, it follows that

$$\begin{aligned} &A_i * A_j = 0, \quad i \neq j, \quad i, j = 1, 2, \ldots, 8, \\ &A_1 + A_2 + \cdots + A_8 \quad \text{is a } (+1, -1)-\text{matrix.} \end{aligned} \tag{7.12}$$

On the other hand, it is not difficult to show that the matrix H_n can also be presented as

$$H_n = \left[(++) \otimes B_1 + (+-) \otimes B_2, \ (++) \otimes B_3 + (+-) \otimes B_4 \right]. \tag{7.13}$$

Now, let us show that matrices B_i satisfy the conditions of Eq. (7.10). From the representation (7.13) and from Eq. (7.12) and $H_n H_n^T = nI_n$, the first three conditions of Eq. (7.10) will follow. Because H_n is a Hadamard matrix of order *n*, then from the representation (7.13), we find the following system of matrix equations:

$$\begin{aligned} &B_1^T B_1 + B_1^T B_2 + B_2^T B_1 + B_2^T B_2 = nI_{n/4}, \\ &B_1^T B_1 - B_1^T B_2 + B_2^T B_1 - B_2^T B_2 = 0, \\ &B_1^T B_3 + B_1^T B_4 + B_2^T B_3 + B_2^T B_4 = 0, \\ &B_1^T B_3 + B_1^T B_4 - B_2^T B_3 - B_2^T B_4 = 0; \end{aligned} \tag{7.14a}$$

$$\begin{aligned} &B_1^T B_1 + B_1^T B_2 - B_2^T B_1 - B_2^T B_2 = 0, \\ &B_1^T B_1 - B_1^T B_2 - B_2^T B_1 + B_2^T B_2 = nI_{n/4}, \\ &B_1^T B_3 + B_1^T B_4 - B_2^T B_3 - B_2^T B_4 = 0, \\ &B_1^T B_3 - B_1^T B_4 - B_2^T B_3 + B_2^T B_4 = 0; \end{aligned} \tag{7.14b}$$

$$\begin{aligned} &B_3^T B_1 + B_3^T B_2 + B_4^T B_1 + B_4^T B_2 = 0, \\ &B_3^T B_1 - B_3^T B_2 + B_4^T B_1 - B_4^T B_2 = 0, \\ &B_3^T B_3 + B_3^T B_4 + B_4^T B_3 + B_4^T B_4 = nI_{n/4}, \\ &B_3^T B_3 - B_3^T B_4 + B_4^T B_3 - B_4^T B_4 = 0; \end{aligned} \tag{7.14c}$$

$$\begin{aligned} &B_3^T B_1 + B_3^T B_2 - B_4^T B_1 - B_4^T B_2 = 0, \\ &B_3^T B_1 - B_3^T B_2 - B_4^T B_1 + B_4^T B_2 = 0, \\ &B_3^T B_3 + B_3^T B_4 - B_4^T B_3 - B_4^T B_4 = 0, \\ &B_3^T B_3 - B_3^T B_4 - B_4^T B_3 + B_4^T B_4 = nI_{n/4}. \end{aligned} \tag{7.14d}$$

They are equivalent to

$$B_i^T B_i = \frac{n}{2} I_{n/4}, \quad i = 1,2,3,4, \\ B_i^T B_j = 0, \quad i \neq j, \quad i = 1,2,3,4. \tag{7.15}$$

Sufficiency: Let $(0,\pm 1)$ matrices $B_i, i = 1,2,3,4$ of dimensions $n \times (n/4)$ satisfy the conditions of Eq. (7.10). We can directly verify that Eq. (7.13) is a Hadamard matrix of order *n*.

Corollary 7.1.1: The (+1,–1) matrices

$$Q_1 = (B_1 + B_2)^T, \qquad Q_2 = (B_1 - B_2)^T, \\ Q_3 = (B_3 + B_4)^T, \qquad Q_4 = (B_3 - B_4)^T \tag{7.16}$$

of dimensions $\frac{n}{4} \times n$ satisfy the conditions

$$Q_i Q_j^T = 0, \quad i \neq j, \quad i = 1,2,3,4, \\ Q_i Q_i^T = nI_{\frac{n}{4}}, \quad i = 1,2,3,4. \tag{7.17}$$

Corollary 7.1.2 [3]: If there be Hadamard matrices of order *n*, *m*, *p*, *q*, then the Hadamard matrix of order *mnpq*/16 also exists.

Proof: According to Theorem 7.1.2, there are $(0,\pm 1)$ matrices A_i and B_i, $i = 1,2,3,4$ of dimensions $m \times (m/4)$ and $n \times (n/4)$, respectively, satisfying the conditions in Eq. (7.10).

Introduce the following (+1,–1) matrices of orders *mn*/4:

$$X = A_1 \otimes (B_1 + B_2)^T + A_2 \otimes (B_1 - B_2)^T, \\ Y = A_3 \otimes (B_3 + B_4)^T + A_4 \otimes (B_3 - B_4)^T. \tag{7.18}$$

It is easy to show that matrices *X*, *Y* satisfy the conditions

$$XY^T = X^T Y = 0, \\ XX^T + YY^T = X^T X + Y^T Y = \frac{mn}{2} I_{mn/4}. \tag{7.19}$$

Again, we rewrite matrices *X*, *Y* in the following form:

$$X=[(++)\otimes X_1+(+-)\otimes X_2,\ (++)\otimes X_3+(+-)\otimes X_4],$$
$$Y=[(++)\otimes Y_1+(+-)\otimes Y_2,\quad (++)\otimes Y_3+(+-)\otimes Y_4]. \tag{7.20}$$

where $X_i,\ Y_i,\ i=1,2,3,4$ are $(0,\pm 1)$ matrices of dimensions $(mn/4)\times(mn/16)$ satisfying the conditions

$$\begin{aligned}
&X_1*X_2=X_3*X_4=Y_1*Y_2=Y_3*Y_4=0,\\
&X_1\pm X_2,\ X_3\pm X_4,\ Y_1\pm Y_2,\ Y_3\pm Y_4 \quad \text{are } (+1,-1)-\text{matrices},\\
&\sum_{i=1}^{4}X_iY_i^T=\sum_{i=1}^{4}X_i^TY_i=0,\\
&\sum_{i=1}^{4}\left(X_iX_i^T+Y_iY_i^T\right)=\sum_{i=1}^{4}\left(X_i^TX_i+Y_i^TY_i\right)=\frac{mn}{4}I_{mn/4}.
\end{aligned} \tag{7.21}$$

Similarly to Hadamard matrices of orders *p* and *q* can be constructed (+1,–1) matrices *P* and *Q* of orders *pq*/4 with the conditions of Eq. (7.19).

Now, consider the following $(0,\pm 1)$ matrices:

$$Z=\frac{P+Q}{2},\qquad W=\frac{P-Q}{2},$$
$$C_i=X_i\otimes Z+Y_i\otimes W,\quad i=1,2,3,4. \tag{7.22}$$

It is not difficult to show that matrices *Z* and *W* satisfy the conditions

$$\begin{aligned}
&Z*W=0,\\
&ZW^T=WZ^T,\\
&ZZ^T=Z^TZ=WW^T=W^TW=\frac{pq}{8}I_{pq/4}.
\end{aligned} \tag{7.23}$$

Assuming that matrices C_i of dimension $(mnpq/16)\times(mnpq/64)$ satisfy the conditions of Eq. (7.10).